Extending a key resource of quantum technology, entanglement—either by having more than two quantum objects or using “qudits” instead of “qubits”—offers prospects for improvements in quantum communications, computation and other technologies.
Quantum technologies have seen a dramatic increase in investment in recent years; in 2020, the sector saw some US$22 billion from public funding alone. With additional private investment from the likes of Google, IBM, Amazon, Alibaba and others, it looks like a quantum revolution is indeed coming sooner than previously expected.

Among the most important of quantum phenomena that will power this revolution is entanglement—the strange connection of the states of two or more quantum particles, even when physically separated from one another. Much experimental work thus far has focused on so-called bipartite, two-dimensional entanglement—the entanglement of one quantum property, such as spin or polarization, that takes on two possible values (a qubit), between two quantum systems (such as individual photons).

Yet quantum scientists and technologists are increasingly interested in higher-dimensional entanglement, involving multiple particles and multiple quantum degrees of freedom. Such entanglement could allow for higher security and greater bandwidth in quantum communications, for example, and enable better error correction for emerging quantum computing platforms. In this feature, we have a look at high-dimensional entanglement, how it’s implemented experimentally, and some of the applications it might advance.

“Spooky action”: An Alice and Bob story
Before diving into high-dimensional entanglement, it is worth a quick review of just what entanglement is.

In a landmark 1935 paper, Albert Einstein, Boris Podolsky and Nathan Rosen (EPR) pointed out that something seemed amiss in the still-young theory of quantum mechanics. At the heart of the problem was that quantum mechanics seemed to allow, as Einstein famously put it in a letter to Max Born, for *spukhafte Fernwirkungen*—“spooky action at a distance”—in that, under the theory, the measurement of the state of one particle could instantaneously define the outcome of the state of the other particle, even if the two particles were far apart.

We can start to understand the strangeness of this by considering a simple thought experiment. Imagine, in the classical (non-quantum) world, that two balls of the same color—red or blue—are each placed in a closed box. One box is then given to Alice and the other to Bob, who open their boxes in separate rooms. The moment Alice sees a blue (or red) ball inside her box, she knows that Bob also has a blue (or red) ball, even without communicating with Bob, and vice versa. There is nothing extraordinary about this correlation, or about the objects involved.

The story is very different in the strange world of quantum entanglement. Quantum objects exist in a superposition of states—the balls, as quantum objects, could be “both blue and red” at the same time, until either Alice or Bob looks inside the box. At that instant, the ball becomes either blue or red, according to the probabilities involved. Color here is a quantum bit with two possible outcomes. Further, if the balls are entangled, then if Alice sees that her ball is blue, Bob will also see that his ball is blue when he opens the box—even though both balls were in an indeterminate state until Alice made her measurement. Apparently, the observation of a blue ball in one room depends on the observation of a blue ball in another room.

This is the crux of entanglement. Alice and Bob observe the same correlations they observed in the classical experiment, but now the origin of the correlation is quantum superposition, not the well-defined (classical) colors of the balls. Moreover, if Alice and Bob were to repeat the experiment, they could get different results. Whether they see blue or red is totally random, but always correlated in that the colors are the same.

The possibility of such counterintuitive phenomena led Einstein, Podolsky and Rosen to conclude that quantum mechanics is not a complete theory of the world—that “local hidden variables” must be involved to explain the apparent entanglement of distant quantum particles. Yet experiment after experiment has shown that quantum entanglement is real. The inherent randomness of quantum mechanics gives rise to correlations that are fundamentally different from the ones we know in the classical world.

Quantum scientists and technologists are increasingly interested in high-dimensional entanglement, involving multiple particles and multiple degrees of freedom.
Complementarity and entanglement

The scenario described above involves quantum correlations between two parties (bipartite) in two dimensions (qubit)—this is bipartite qubit entanglement. Entanglement involving more than two parties is called multipartite entanglement. Entanglement of properties that could take on $d$ values—so-called "quantum dits"—is called qudit entanglement.

When more than two parties share qubits or two or more parties share qudits, we enter the realm of high-dimensional entanglement. And to fully appreciate high-dimensional entanglement, we first must understand another property of quantum systems, complementarity.

First conceived of by Niels Bohr in 1927, complementarity is the principle that quantum objects have complementary properties that cannot be measured with certainty at the same time. This is perhaps most famously captured in Werner Heisenberg’s uncertainty principle, which holds that the position and momentum of a free particle can't be measured with the same certainty simultaneously. Other examples might include the amplitude and phase of a photon, or any of a range of complementary properties.

From the point of view of the thought experiment laid out above, imagine that the objects in the boxes given to Alice and Bob can experience not only correlations in color (red or blue), but also in shape (balls or cubes). In the classical world, Alice and Bob can measure the color and shape of their object simultaneously. In the quantum world, if color and shape were complementary quantum properties, then if Alice were to measure the color of her object and find that it is blue, measuring the shape afterwards would result in a random result (ball or cube), as if the information on shape has been lost.

Entanglement and complementarity are intertwined. Entanglement enforces that no single-object properties (such as color or shape above) can be predicted with certainty for quantum objects; complementarity gives us a way of detecting entanglement. If $A_1$ and $A_2$ are complementary measurable properties of object $A$, the uncertainties of $A_1$ and $A_2$ cannot both be zero; the same holds for object $B$ and its properties $B_1$ and $B_2$. Joint measurements ($J$) of complementary properties of $A$ and $B$—defined as $J = A_1 \otimes I + I \otimes B_1$, where I is the identity operation, $i = [1,2]$ and $\otimes$ denotes the tensor product—could have uncertainties that are simultaneously zero, but only if $A$ and $B$ are entangled. This means that the

An Alice and Bob Story

**Entanglement**

In the classical world ...

Both Alice and Bob receive a closed box with identical, quantum-entangled balls that are of indeterminate color —“both red and blue” (quantum superposition)

When Alice opens the box and looks in—“measuring” the ball’s color—she will see only one of the superposed colors (red in this case)

Because the ball in Bob’s box is entangled with Alice’s, once she has made her measurement, Bob, in the next room, will see a ball of the same color that Alice measured

In the quantum world ...

If Alice and Bob—each with a box containing a single ball of the same color [red or blue]—go into separate rooms

If Alice, on opening her box, finds a blue ball ...

... she knows that Bob also has a blue ball, whether he has opened his box or not

If Alice instead first measures the object’s color (e.g., blue) ...

... then later measures shape, Alice will find a blue object but the shape might be either a ball or a cube

If Alice opens her box and first measures the object’s color [e.g., blue] ...

... then later measures shape, Alice will find a blue object but the shape might be either a ball or a cube

If Alice instead first measures the object’s shape (e.g., cube) ...

... then later measures color, Alice will find a cube but the color might be either red or blue
violation of some local uncertainty relations can be used to detect entanglement.

**Multipartite entanglement: GHZ states**

With the discussion above as background, we can now explore high-dimensional entanglement. One obvious way that such entanglement can be manifested is when the entanglement involves three or more quantum objects. One example of such multipartite entanglement is Greenberger-Horne-Zeilinger (GHZ) states, first studied in detail in 1989. The simplest GHZ state consists of three objects that are all mutually entangled—observed experimentally in 1999 via the entanglement of the polarization property in a three-photon system.

The GHZ state is an example of an absolutely maximally entangled (AME) state—states that exhibit the most entanglement, regardless of how the particles are partitioned. As the number of particles increases, entanglement naturally becomes richer. For example, it is known that three entangled particles can be transformed such that they fall into any of two families of entangled states (of which one is the GHZ family and the other is the so-called W family). For four qubits, there are nine families—that is, two- or three-particle entanglement, distributed among the four particles. However, there does not exist an AME state for four qubits, nor for seven qubits; so far we know that AME states exist only for \( N = 2, 3, 5 \) or 6.

More than a theoretical curiosity, multipartite entanglement, as in the AME states, is a resource for various applications. It can be used for quantum teleportation, allowing information to be sent from one party to another via only classical communication, consuming the entanglement. It can also be used for quantum secret sharing, when quantum information needs to be shared with several parties—not all of whom could be trusted. It could be used in quantum error correction as well—to increase the robustness in quantum computations, for example, as the multiple particles could offer redundancy against erasure errors. And it may help to improve the sensitivity in phase estimation beyond the shot-noise limit, important for metrology applications.

**From qubits to qudits**

The promising applications listed above, plus the complex structure and inherent interest of multipartite entanglement, have motivated plenty of theoretical and experimental investigations. But multipartite entanglement is challenging to achieve in photons, which do not interact strongly with one another. Experiments in high-dimensional entanglement thus have often taken another form: bipartite entanglement (entanglement of two particles) involving multiple degrees of freedom (DOFs). This moves us from the realm of qubits to qudits—with the number of levels necessary to describe the quantum correlation often referred to as the dimensionality of the entanglement (the \( d \) in qudit).

The dimension in this case can be thought of as referring to the size of the relevant “alphabet.” A familiar analogy is the correlations in our DNA. The immense diversity of life is written in a 4D alphabet—the bases adenine (A), thymine (T), guanine (G) and cytosine (C), which pair in a specific way in the DNA double helix. However nature stumbled upon this four-letter alphabet, it accounts for the complexity and diversity of life that we observe in the natural world. It is quite impressive what you can write using just these four letters!
Similarly, entanglement of multiple DOFs offers a rich alphabet for entanglement-based quantum technology. How does one create such qudit entanglement in photons? One way is spontaneous parametric down-conversion (SPDC), in which a high-energy pump photon is converted (with a finite probability), through light–matter interactions in a nonlinear crystal, into two lower-energy “signal” and “idler” photons. Conservation of energy and momentum ensures that the photon pair has properties that are entangled. From energy conservation, we get a frequency-matching condition ensuring that the sum of the energies of the signal and idler photons equals that of the pump photon. This leads to frequency entanglement—the correlation of the frequency of one photon with that of the other. From momentum conservation, meanwhile, we get a phase-matching condition that leads to the entanglement of the transverse spatial mode—or shape—of the photons. Orbital angular momentum (OAM) is one well-known example: light that has an azimuthal phase dependence of $\ell \phi$—where $\phi$ is the azimuthal angle and $\ell$ is an integer—carries an OAM of $\ell \hbar$ per photon. The process of SPDC in a thin crystal conserves OAM; the sum of the OAM of the signal and idler photons is equal to that of the pump photon, and this leads to OAM entanglement. Thus the process of SPDC naturally leads to bipartite high-dimensional entanglement, because both frequency and OAM can be used as qudits—bipartite qudit entanglement comes for free. We can exploit entanglement of multiple degrees of freedom—referred to as hyper-entanglement—and this can be engineered via the SPDC phase-matching.

**Verifying qudit entanglement**

While the ability to create entangled qudits in photons is interesting in itself, to make it useful we must be able to certify that such entanglement exists. Indeed, the first experimental efforts in photonic high-dimensional entanglement focused on just such certification. There are several ways to verify high-dimensional entanglement.

**Quantum state tomography**

In a computed-tomography (CT) scan, X-rays coming from different directions allow clearer visualization than is possible from any single direction. Similarly, one can reconstruct the quantum state by doing multiple “tomographic measurements” in a quantum experiment. The result of such quantum state tomography (QST) is the density matrix, the mathematical object that fully describes the quantum state. The density matrix can then be used to check entanglement mathematically—specifically, if the partial transpose of the matrix (with respect to one party) has negative eigenvalues, the system is entangled.

The first QST for an entangled qutrit ($d = 3$) was done in 2004, on the transverse shape of photons generated from SPDC. Since then, larger and larger density matrices have been reconstructed. Yet significant progress is hindered by the fact that QST does not scale favorably with dimension—that is, for an $n$-particle entangled qudit state, the density matrix has $d^{2n} - 1$ parameters.

**Entanglement witnesses**

Because of the poor scaling of QST, other means have been devised to check for entanglement that do not...
rely on knowing the density matrix. These are called entanglement witnesses. An entanglement witness is a functional that distinguishes entangled from non-entangled states. To be useful, it should lead to a number that can be obtained experimentally.

One familiar example is the $S$-parameter in the Bell inequality, which can be calculated from counting correlations. If $|S| > 2$, the correlations observed cannot be explained by any local deterministic theory. For any pure two-particle, two-dimensional state, this violation also means that the two particles are entangled. Hence the value of $S$ is one possible entanglement witness.

The experimental violations of the Bell inequality published in the 1970s and 1980s were landmark works, but were limited to two dimensions. The Bell inequality have since been extended to higher dimensions in the Collin-Gisin-Linden-Massar-Popescu (CGLMP) inequality. Higher dimensions provide stronger violations, and these have been experimentally demonstrated—for example, up to 4 dimensions using time bins and 11 dimensions using OAM.

While the Bell inequality is useful as a potential entanglement witness, it is worth emphasizing that to violate a Bell inequality, entanglement is necessary but not sufficient. A Bell violation is, strictly speaking, a hallmark of nonlocality, which in itself is a different resource from entanglement. Nevertheless, Bell inequality violations figure greatly in certifying entanglement in “device-independent” scenarios—that is, even without the characterization of specific devices. This is especially important in realistic quantum communications, where there could be untrusted parties.

Certifying entanglement—both verifying that there is entanglement and putting a lower bound on its dimensionality—remains an active area of research. Recent developments have greatly removed the required assumptions and number of measurements, pointing to increasing practicality and efficiency for verifying high-dimensional entanglement. That will be important as we move toward harnessing the advantages of such entanglement in real applications.

Some applications of high-dimensional entanglement

Quantum communications

In communications, high-dimensional entanglement clearly brings a higher capacity for quantum information—one can send more information per photon if one uses multiple degrees of freedom in an entangled qudit. Higher dimensions also bring more security by making it harder for an eavesdropper to go undetected.

In a quantum key distribution (QKD) scenario where Alice sends photons to Bob, the simplest possible attack is an intercept-and-resend strategy. Eve, the eavesdropper, could intercept the photon from Alice and prepare another photon to send to Bob, based on the outcome of her measurements. In doing so, however,
In communications, high-dimensional entanglement brings a higher capacity for quantum information—one can send more information per photon if one uses multiple degrees of freedom.

Eve introduces an error that increases as the number of outcomes increase. Thus higher dimensions make it easier for Eve’s nefarious acts to be sensed.

Entanglement-based QKD has been demonstrated using qudit DOFs in photons. For example, information encoded in the arrival times of single photons (time-bin) has the advantage of being well-preserved even as the photon travels through an optical fiber. A secure key rate of 2.7 Mbit/s has been achieved through 20 km of fiber; 4D entanglement in time-bins has also been demonstrated through 100 km of fiber, although with a lower key rate. Frequency is another DOF that is suitable for standard optical fibers, although the required dispersion cancellation makes scaling up to higher dimensions difficult. Nevertheless, frequency-bin entanglement has been demonstrated over 24 km of fiber.

Spatial DOFs are challenging to transmit through standard optical fibers. New fiber optic technologies such as those that support multiple transverse spatial modes or that have multiple cores are usually needed, although a number of groups have recently demonstrated high-dimensional OAM entanglement through kilometer-scale lengths of fiber. Quantum communication will inevitably involve free-space links, and here spatial modes hold promise; four-dimensional entanglement distribution over a 3-km free-space intracity link has been demonstrated. A practical high-dimensional entanglement-based QKD in free space will need a probe beam and active compensation at a fast rate to reverse the effect of atmospheric turbulence.

Quantum computation

Entanglement plays a role in both circuit-based and measurement-based models of quantum computation, but the extent and nature of that role are still being explored. Curiously, for measurement-based quantum computation—which involves massive “cluster states” of interconnected entangled qubits—more entanglement does not imply more computational power. Measurement-based qudit quantum computing, where the initial cluster state comprises qudits rather than qubits, is relatively unexplored, although three- and four-level cluster states have been experimentally realized in the time and frequency domain.

In the quantum circuit model, using qudits leads to some advantages, including reduced circuit complexity and enhanced algorithm efficiency. However, the combined challenges of making an efficient photon source, experimentally implementing efficient high-dimensional gates, and minimizing photon loss has prevented a definitive demonstration of these advantages.

Recent developments of on-chip technologies should accelerate high-dimensional photonic quantum computation. In particular, linear optics based on silicon photonics have achieved 15×15 high-dimensional entanglement using photon path. The stretch goal is to show a genuine quantum computational advantage over both classical and qubit-based quantum computation.

Ready to level up?

Photons are ideal carriers of quantum information—especially for applications that require transfer of information in free space or through optical fiber. A photon is a rich physical system with several high-dimensional properties, which become entangled for free via SPDC.

Although this suffices for basic studies of bipartite high-dimensional entanglement, practical applications will require efficient sources of multiphoton entanglement, manipulation techniques and multilevel detection schemes. Theoretical investigations are motivating the development of these technologies, and we are already seeing compelling arguments for using high-dimensional entanglement—but the complexity of theory and experiment hinders quick progress.

Nevertheless, Hilbert space is a big space indeed—and we believe that future excursions will continue to demonstrate the benefits of high-dimensional entanglement, and move it toward real-world applications.

The authors thank Kaumudibikash Goswami and Andrew White for a careful reading of this article.

Jacqueline Romero [m.romero@uq.edu.au] and Markus Rambach are with the University of Queensland, Brisbane, Australia.
### References and Resources